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CUTTING UP GRAPHS

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Let Γ be infinite connected graph with more than one end. It is shown that there is a subset $d \subset V\Gamma$ which has the following properties. (i) Both d and $d^* = V\Gamma \setminus d$ are infinite. (ii) There are only finitely many edges joining d and d^*. (iii) For each $g \in \mathrm{Aut}\,\Gamma$ at least one of $d \subset dg$, $d^* \subset dg$, $d \subset d^*g$, $d^* \subset d^*g$ holds. Any group acting on Γ has a decomposition as a free product with amalgamation or as an HNN-group.

1. Introduction

Let Γ be an infinite connected graph. Note that it is not assumed that Γ is locally finite. The number $e(\Gamma)$ of ends of Γ is defined to be the least upper bound (possibly ∞) of the number of connected components with infinitely many vertices that can be obtained by removing finitely many edges. Thus $e(\Gamma) > 1$ if and only if there is a subset $c \subset V$ such that both c and $c^* = V\Gamma \setminus c$ are infinite and

$$\delta c = \{e \in E\Gamma \mid \text{ one vertex of } e \text{ lies in } c \text{ and one in } c^*\}$$

is finite.

A subset $c \subset V\Gamma$ such that δc is finite is called a *cut*. Note that this is not standard terminology. A cut is said to be *non-trivial* if both c and c^* are infinite. If c, d are cuts then so are $c \cap d$ and $c \cup d$. In this paper we prove the following.

Theorem 1.1. *Let Γ be a graph with more than one end. There exists a non-trivial cut $d \subset V\Gamma$ such that for any $g \in \mathrm{Aut}\,\Gamma$ one of the inclusions*

$$d \subset dg, \quad d \subset d^*g, \quad d^* \subset dg, \quad d^* \subset d^*g$$

holds.

As an application it is shown that if G is any group acting on Γ, then G has a decomposition $G = A *_C B$ as a free product with amalgation or as an HNN group $G = \mathrm{gp}\,(A, t \mid t^{-1}Ct = D)$ where C, D are isomorphic subgroups of A and in both cases C is the stabilizer of d and each of the stabilizers of the directed edges of δd is a subgroup of finite index in C. This decomposition will be a proper decomposition, i.e. if $G = A *_C B$ then $C \neq A$ and $C \neq B$, if Γ/G is finite. This generalizes the Stallings

AMS subject classification (1980): 05 C 25, 20 F 05 structure theorem [5.5A9]. More generally the decomposition is proper if and only if there exists $g \in G$ such that dg properly contains d.

Theorem 1.1 can be used to derive results about splitting an n-manifold along an $(n-1)$-manifold [1]. It has also been used by H. D. Macpherson to classify infinite distance transitive graphs with finite valency [4].

2. Narrow cuts

Let Γ be a connected graph with more than one end so that non-trivial cuts exist. Let c be a cut. Let $w(c)=|\delta c|$. Let k be the minimal value of $w(c)$ for c non-trivial. A cut is called narrow if $w(c)=k$. The following statements are proved in [5, p. 50—51].

2.1. *Let $c_1 \supset c_2 \supset \dots$ be a descending sequence of narrow cuts. Suppose*

$$b = \bigcap_{n=1}^{\infty} c_n \neq \emptyset,$$

then $b=c_n$ for some n. ∎

2.2. *There is a narrow cut c which is minimal with respect to containing a fixed $v \in V\Gamma$. That is if b is narrow and $v \in b \subset c$ then $b=c$.* ∎

2.3. *If c is narrow and minimal with respect to containing $v \in V\Gamma$ and b is a narrow cut then at least one of $c \cap b$, $c \cap b^*$, $c^* \cap b$, $c^* \cap b^*$ is finite.* ∎

Part of the following result is also obtained by Stallings [5, p. 52].

2.4. *Let b, c be narrow cuts such that $b \cap c$ and $b^* \cap c^*$ are both infinite. Then $b \cap c$ and $b^* \cap c^*$ are narrow. Exactly half the edges of $b \cap c^*$ lie in δb and the other half lie in δc. No edge of $\delta(b \cap c^*)$ lies in both δb and δc.*

Proof. In Fig. 1 the Greek letters indicate the number of edges of $E\Gamma$ connecting the various pairs of sets as shown. Thus for instance there are ϱ edges with one vertex in $b \cap c$ and one vertex in $b^* \cap c^*$.

From Fig. 1, we see that

$$k = w(b) = \alpha + \varrho + \tau + \gamma$$

$$k = w(c) = \nu + \varrho + \tau + \beta.$$

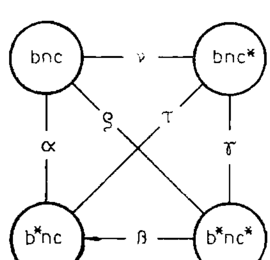

Fig. 1

We know that $b\cap c$ and $b^*\cap c^*$ are non-trivial cuts. Hence

$$k \leqq w(b\cap c) \quad = \alpha+\varrho+\nu$$

and

$$k \leqq w(b^*\cap c^*) = \beta+\varrho+\gamma.$$

Hence

$$\tau+\gamma \leqq \nu \quad \text{and} \quad \nu+\tau \leqq \gamma.$$

It follows easily that $\tau=0$ and $\gamma=\nu$, as required. ∎

Let N be the set of narrow cuts in Γ.

2.5. *Let $e\in E\Gamma$. There are only finitely many $b\in N$ such that $e\in\delta b$.*

Proof. Let v be one of the vertices of e. Let

$$N_1 = \{b\in N \mid v\in b \text{ and } e\in\delta b\}.$$

Let $b, c\in N_1$. Suppose $b\cap c$ is finite, then $b\cap c^*$ and $b^*\cap c$ are both infinite. By 2.4 no edge of $\delta(b\cap c)$ lie in both δb and δc. But $e\in\delta b\cap\delta c\cap\delta(b\cap c)$ which is a contradiction. Thus $b\cap c$ is infinite. Similarly $b^*\cap c^*$ is infinite. Again it follows from 2.4 that $b\cap c$ and $b^*\cap c^*$ are narrow. Note that $b^*\cap c^*=(b\cup c)^*$. Thus $b\cup c$ is narrow. It can be seen that N_1 is a lattice. It follows from 2.1 that N_1 has unique minimal element d_0. Let $N_1^*=\{b^*|b\in N_1\}$. Each element of N_1^* contains w, the other vertex of e. Hence N_1^* contains a unique minimal element and so N_1 contains a unique maximal element d_1. Choose a maximal chain $d_0=b_1\subset b_2\subset\ldots\subset b_m=d_1$ of elements of N_1. Such a chain exists by 2.1. Let $b\in N_1$. Let $r=\max\{i|b_i\subset b\}$ and $s=\min\{j|b\subset b_j\}$. We show by induction on $s-r$ that every edge of δb is contained in δb_i for some i. If $s-r=0$, then $b=b_r=b_s$. If not $b_r\subset b\subset b_s$. Let $b'=b\cup b_{r+1}$. Then by the induction hypothesis, every edge of $\delta b'$ is contained in δb_i for some i. Also $b\cap b_{r+1}=b_r$ and $b\cup b_{r+1}=b'$ and it can be seen from Fig. 1 that every edge of δb lies in at least one of δb_r, δb_{r+1} or $\delta b'$. Hence every edge of δb lies in some δb_i. It follows that there are only finitely many possibilities for δb. But a cut b is determined by δb together with an orientation of the edges of δb. Thus there are only finitely many possibilities for b. ∎

2.6. *Let $c\in N$. There are only finitely many $b\in N$ which do not satisfy one of*

$$b\subset c, \quad b^*\subset c, \quad b\subset c^*, \quad b^*\subset c^*.$$

Proof. Let F be a finite connected subgraph of Γ containing each edge of δc. Suppose $\delta b\cap EF=\emptyset$. Since F is connected this means that $VF\subset b$ or $VF\subset b^*$. Suppose that $VF\subset b$. Referring back to Fig. 1 we see that $\varrho=\tau=\beta=0$ and so $\nu=k$. Also $\alpha+\gamma=k$. Since one of $b^*\cap c$ and $b^*\cap c^*$ must be a non-trivial cut, it follows that either $\alpha=k$ and $\gamma=0$ or $\alpha=0$ and $\gamma=k$. Thus $b^*\cap c=\emptyset$ or $b^*\cap c^*=\emptyset$, i.e. $b^*\subset c^*$ or $b^*\subset c$. If $VF\subset b^*$ then in a similar manner it follows that $b\subset c^*$ or $b\subset c$. Finally it follows from 2.5 that there are only finitely many $b\in N$ for which $\delta b\cap EF\neq\emptyset$. ∎

Write $b\subset^a c$ if $b\setminus c$ is finite. Write $b=^a c$ if $b\subset^a c$ and $c\subset^a b$. Let $\hat{c}= =\cap\{b\in N|c\subset^a b\}$.

2.7. *Either* $\hat{c}=\emptyset$ *or* $\hat{c}={}^a c$ *and* $\hat{c}\in N$.

Proof. If $c\subset{}^a b$ then $b\cap c\in N$ and $b\cap c={}^a c$. Thus $\hat{c}=\bigcap\{b\in N|b={}^a c\}$. But $N_c=\{b\in N|b={}^a c\}$ is a lattice by 2.4. By 2.1 if $\hat{c}\neq\emptyset$ then $\hat{c}$ is the unique minimal element of N_c. ▮

Let $c\in N$ be a cut for which $\hat{c}\neq\emptyset$ and $\widehat{c^*}\neq\emptyset$. For any $v\in V\Gamma$ let

$$A(v, c) = \{b\in N\,|v\in b\subset{}^a c\}.$$

2.8. *The set* $A(v, c)$ *is finite.*

Proof. If $b\subset{}^a c$ then $c^*\subset{}^a b^*$. Hence $\widehat{c^*}\subset b^*$ and $b\subset(\widehat{c^*})^*=d$. Let F be a connected finite subgraph of Γ containing v and an edge of δd. Since $v\in b\subset d$, δb must contain an edge of F. By 2.5 there are only finitely many such b. ▮

Let $\alpha(v, c)=|A(v, c)|$. Put

$$c^0 = \{v|\alpha(v, c) > \alpha(v, c^*)\}.$$

Let $\bar{c}=((c^*)^0)^*$. Thus if $v\in\bar{c}$, $v\notin(c^*)^0$. Hence $\alpha(v, c^*)\leqq\alpha(v, c)$, i.e.

$$\bar{c} = \{v|\alpha(v, c) \geqq \alpha(v, c^*)\}.$$

We see that $c^0\subset\bar{c}$.

2.9. *The following inclusions are satisfied,*

$$\hat{c}\subset c^0 \subset\bar{c}\subset(\widehat{c^*})^*.$$

Also $c^0={}^a c$.

Proof. Let $v\in\hat{c}$. Suppose $b\in A(v, c^*)$. Then $b\subset{}^a c^*$, i.e. $c\subset{}^a b^*$. But this means that $\hat{c}\subset b^*$ and so $v\notin b$ which is a contradiction. Thus $\alpha(v, c^*)=0$. However $\hat{c}\in A(v, c)$ and so $\alpha(v, c)\geqq 1$. Thus $\hat{c}\subset c^0$. Similarly $\widehat{c^*}\subset(c^*)^0$ and hence $\bar{c}\subset(\widehat{c^*})^*$. But $\hat{c}={}^a c$ and $(\widehat{c^*})^*={}^a c$ by 2.7. Therefore $c^0={}^a c$ and the proof of 2.9 is complete. ▮

Let $b\in N$ be another cut for which $\hat{b}\neq\emptyset$ and $(\widehat{b^*})^*\neq\emptyset$. Note that if $g\in\operatorname{Aut}\Gamma$ then $b=cg$ will have this property.

2.10. *If* $c\subset{}^a b$ *then* $c^0\subset b^0$ *and* $\bar{c}\subset\bar{b}$. *If* $c\subset{}^a b$ *but* $c\neq{}^a b$, *then* $\bar{c}\subset b^0$.

Proof. If $c\subset{}^a b$ then $A(v, c)\subset A(v, b)$ and $A(v, b^*)\subset A(v, c^*)$. Thus $\alpha(v, c)\leqq\alpha(v, b)$ and $\alpha(v, b^*)\leqq\alpha(v, c^*)$. It follows immediately that $c^0\subset b^0$ and $\bar{c}\subset\bar{b}$. If $c={}^a b$ then $\hat{c}=\hat{b}$ and $c^0=b^0$. Suppose $\alpha(v, c)=\alpha(v, c^*)$, that is $v\in\bar{c}\setminus c^0$. By 2.9 $v\notin\hat{c}$. Now $\widehat{c^*}\in A(v, c^*)$ and $\widehat{c^*}\notin A(v, b^*)$ if $\hat{b}$ properly contains $\hat{c}$. Thus $\alpha(v, b^*)<\alpha(v, c^*)$. Hence $v\in b^0$. It follows that $\bar{c}\subset b^0$ if $c\subset{}^a b$ but $c\neq{}^a b$. ▮

3. Proof of Theorem 1.1

We know from 2.2 that there exists $c\in N$ such that for any $b\in N$ one of $b\cap c$, $b^*\cap c$, $b\cap c^*$, $b^*\cap c^*$ is finite.

Case 1. $\hat{c}\neq\emptyset$, $\widehat{c^*}\neq\emptyset$.

3.1. *Let* $g \in \text{Aut}\,\Gamma$ *and* $d = c^0$ *then at least one of* $d \cap dg$, $d^* \cap dg$, $d \cap d^*g$, $d^* \cap d^*g$ *is empty.*

Proof. Suppose $c \subset^a cg$, then $c^0 \subset (cg)^0$. But $(cg)^0 = c^0 g$. Hence $d \subset dg$. Similarly if $c^* \subset^a c^* g$ then $\overline{c^*} \subset \overline{c^* g}$. However $\overline{c^*} = (c^0)^* = d^*$ and so $d^* \subset d^* g$. If $c^* =^a cg$, then $(cg)^0 \subset \overline{c^*}$ by 2.9. Thus $dg \subset d^*$. If $c^* \subset^a cg$ but $c^* \neq^a cg$ then, by 2.10, $\overline{c^*} \subset (cg)^0$ and so $d^* \subset dg$. If $c \subset^a c^* g$, then $c^0 \subset \overline{c^* g}$ and so $d \subset d^* g$. ▮

Case 2. $\hat{c} = \emptyset, \widehat{c^*} = \emptyset$.

3.2. *For every* $b \in N$ *either* $b =^a c$ *or* $b =^a c^*$.

Proof. Suppose $b \subset^a c$. We know that the set $N_c = \{d \in N | d =^a c\}$ is closed under finite intersections and that $\bigcap \{d \in N_c\} = \emptyset$. Now N_c is infinite and so it contains a cut d for which one of

$$d \subset b, \quad d^* \subset b, \quad d \subset b^*, \quad d^* \subset b^*$$

holds by 2.6. We can also assume that d does not contain a particular vertex of b and so $d^* \subset b^*$ is impossible. Also $b \subset^a c =^a d$, and so $d \subset b^*$ and $d^* \subset b$ are impossible. Thus $d \subset b$ and $b =^a c$. The other cases ($b^* \subset^a c$, $b \subset^a c^*$, $b^* \subset^a c^*$) are all treated similarly. ▮

In fact it is fairly easy to show that Γ has 2 ends, i.e. for every nontrivial cut b either $b =^a c$ or $b =^a c^*$.

If $b, d \in N$ write $b \leqq d$ if $b =^a b$ and $|b \setminus d| \leqq |d \setminus b|$. It is easy to show that $\leqq$ is transitive. There may be distinct cuts b, d for which $b \leqq d$ and $d \leqq b$.

Let $G = \text{Aut}\,\Gamma$. Let $H = \{h \in G | ch =^a c\}$. Now H is a subgroup of G and $(G:H) \leqq 2$. Let

$$H^- = \{h \in H | ch \leqq c\}$$

and

$$H^+ = \{h \in H | c \leqq ch\}.$$

It is easy to see that H^- and H^+ are closed under multiplication.
Let

$$d = \bigcup \{ch | h \in H^-\}$$

3.3. *The subset* d *is a narrow cut and* $d =^a c$.

Proof. It follows from 2.6 that the set

$$S = \{b \in N | b \leqq c \text{ and } b \not\subset c\}$$

is finite. But $d = c \cup \{ch \in S | h \in H\}$, Thus d is the union of finitely many cuts ch for each of which $ch =^a c$. It follows from 2.4 that $d \in N$ and $d =^a c$. ▮

3.4. *If* $h \in H$, *then either* $dh \subset d$ *or* $d \subset dh$

Proof. If $h \in H^-$ then $dh \subset d$. If $h \in H^+$ then $h^{-1} \in H^-$ and so $dh^{-1} \subset d$, i.e. $d \subset dh$. ▮

If $b =^a c$ and $ch \leqq c$ then $bh \leqq b$. To see this note that $ch \leqq c$ means that $\int (c - ch) \geqq 0$. Here we identify a set with its characteristic function. For any function $f: V\Gamma \to Z$ which has finite support $\int (f - fh) = 0$. But $b - c$ has finite support and so $\int (c - ch) = \int (b - bh)$.

Suppose $G \neq H$. Let $x \in G \setminus H$ and let $h \in H$. Now

$$\begin{aligned}\int (c - cxhx^{-1}) &= \int (cx - cxh) \\ &= \int (c^* - c^* h) \\ &= -\int (c - ch).\end{aligned}$$

Thus if $h \in H^+$, then $xhx^{-1} \in H^-$. But $x^2 \in H$ and x^2 commutes with x and so $x^2 \in H^+ \cap H^-$. Thus $dx^2 = d$. If $dh = d$ for every $h \in H$ put $d_1 = d \cap d^* x$. Now $xh = h'x$ for some $h' \in H$. Hence $d^* xh = d^* x$ and $d_1 h = d$. Also $d_1 x \cap d_1 = \emptyset$. Thus for any $g \in G$ either $d_1 g = d_1$ or $d_1 g \cap d_1 = \emptyset$.

Suppose finally that there exists $h \in H$ such that $dh \neq d$. Choose $y \in H^+$ so that $dy - d$ has the smallest possible non-zero number of elements. Put $d_i = dy^i$. If $g \in G$ then $dg = d_i$ or $d_i x$ for some $i \in \mathbf{Z}$. Let $u_i = d_i \cap d_i^* x$ and let

$$u = \bigcup \{u_i | i \in \mathbf{Z}\}.$$

First we show that $u \cap ux = \emptyset$. Now $ux = \bigcup \{u_i x | i \in \mathbf{Z}\}$. It suffices to show that $u_i \cap u_j x = \emptyset$ for any i, j. Since $u_i x^2 = u_i$ we can assume $i \leqq j$. But

$$u_i \cap u_j x = d_i \cap d_i^* x \cap d_j^* \cap d_j x = \emptyset$$

since $d_i \subset d_j$. Next we show that $(ux)^* = u^* x \subset uy$. Now

$$\begin{aligned}u^* x &= \bigcap \{u_i^* x | i \in \mathbf{Z}\} \\ &= \bigcap \{(d_i^* \cup d_i x) x | i \in \mathbf{Z}\} \\ &= \bigcap \{d_i^* x \cup d_i | i \in \mathbf{Z}\}.\end{aligned}$$

Let $v \in u^* x$. We know that $d_i \subset d_j$ if $i \leqq j$ and $\bigcap \{d_i | i \in \mathbf{Z}\} = \emptyset$ and $\bigcup \{d_i | i \in \mathbf{Z}\} = V\Gamma$. Thus there is a smallest integer m such that $v \in d_m$ but $v \notin d_{m-1}$. However $v \in d_{m-1}^* x \cup d_{m-1}$ and so $v \in d_{m-1}^* x$. Next note that $d_i xy = d_i y^{-1} x = d_{i-1} x$. Thus

$$\begin{aligned}uy &= \bigcup \{u_i y | i \in \mathbf{Z}\} \\ &= \bigcup \{d_i y \cap d_i^* xy | i \in \mathbf{Z}\} \\ &= \bigcup \{d_{i+1} \cap d_{i-1}^* x | i \in \mathbf{Z}\}.\end{aligned}$$

Now $v \in d_m \subset d_{m+1}$ and so $v \in d_{m+1} \cap d_{m-1}^* x$. Hence $v \in uy$. Thus $u^* x \subset uy$.

From these inclusions we obtain the sequence

$$u \subset u^* x \subset uy \subset u^* xy \subset uy^2 \subset u^* xy^2 \subset \ldots.$$

It follows that for any $g \in G$ one of the inclusions

$$u \subset ug,\ u^* \subset ug,\ u \subset u^* g,\ u^* \subset u^* g$$

holds.

Case 3. $\hat{c} = \emptyset, \widehat{c^*} \neq \emptyset$.

As in 3.2 it can be shown that if $b \in N$ and $b \subset^a c$ then $b =^a c$. Write $b \sim d$ if $|b \setminus d| = |d \setminus b|$ and $b =^a d$. Thus $b \sim d$ if $b \leqq d$ and $d \leqq b$.

There is no $g\in G$ for which $c^*g\subset^a c$, since this would mean that $\widehat{c^*}=\emptyset$. Suppose $c\subset^a cg$ then $cg^{-1}\subset^a c$ and so $c=^a cg$. If $cg\geqq c$ but not $cg\sim c$ then $|cg^k\setminus c|$ can be made arbitrarily large by choosing k large enough. But $cg^k\subset(\widehat{c^*})^*=^a c$. Hence $cg\leqq c$. But then $c\leqq cg^{-1}$ and we again have a contradiction unless $c\sim cg$. Similarly if $cg\subset^a c$ then $c\sim cg$. Thus for each $g\in G$ either $cg\sim c$ or $cg\subset^a c^*$. Let $N'=\{cg, c^*g|g\in G\}$. Put

$$d = \cap\{b\in N'|c\subset^a b\}.$$

Using 2.6 it can be shown that there are only finitely many $b\in N'$ for which $c\subset^a b$ but not $c\subset b$. Thus $d\in N$ by 2.4 and $d=^a c$. If $cg\sim c$ then the action of g permutes the set $\{b\in N'|c\subset^a b\}$. Hence $dg=d$. If $cg\subset^a c^*$ then $c\subset^a c^*g^{-1}$ and so $d\subset c^*g^{-1}$. But $d\subset c$ and so $dg\subset c^*\subset d^*$. Thus for any $g\in G$ either $dg=d$ or $dg\subset d^*$.

Case 4. $\hat{c}\neq\emptyset$. $\widehat{c^*}=\emptyset$.

This case can be treated as Case 3 with c^* replacing c.

Every case has now been considered and so the proof of Theorem 1.1 is complete. ∎

4. Applications

Let $(E, \leqq)$ be a partially ordered set with a mapping $E\to E$, $e\to\bar{e}$ for which $\bar{\bar{e}}=e$, satisfying the following conditions

(1) if $e\leqq f$ then $\bar{f}\leqq\bar{e}$
(2) if $e\leqq f$, there are only finitely many $h\in E$ for which $e\leqq h\leqq f$;
(3) for any pair e, f at least one of $e\leqq f$, $e\leqq\bar{f}$, $\bar{e}\leqq f$, $\bar{e}\leqq\bar{f}$ holds;
(4) for no pair e, f is $e\leqq f$ and $\bar{e}\leqq f$.
(5) for no pair e, f is $e\leqq f$ and $e\leqq\bar{f}$.

It is shown in [3, Theorem 2.1] that there exists a tree T for which E is the directed edge set and the order on E is such that if $e\leqq f$ then there is an edge path of directed edges in T

$$e = e_1, e_2, \ldots, e_n = f$$

where the last vertex of e_i is the first vertex of e_{i+1} for $i=1,\ldots, n-1$.

Theorem 4.1. *Let the group G act on the graph Γ and suppose $e(\Gamma)>1$. Then G acts on a tree T in such a way that T/G has one edge. The stabilizer of each edge of T contains the stabilizer of a directed edge of Γ as a subgroup of finite index.*

Proof. By Theorem 1.1 there exists a non-trivial cut $d\subset V\Gamma$ such that for each $g\in G$ one of

$$d\subset dg,\quad d\subset d^*g,\quad d^*\subset dg,\quad d^*\subset d^*g$$

holds. Let $E=\{dg, d^*g|g\in G\}$. Put $\bar{e}=e^*$ for $e\in E$. Clearly $(E, \subset)$ satisfies conditions (1), (3), (4) and (5) above. We need also to show that condition (2) is satisfied. If d is narrow then condition (2) is satisfied using 2.2. It can be seen from the proof of Theorem 1.1 that d is narrow in Cases 2, 3 and 4. In Case 1 it can also be shown that d is narrow. However it is not necessary to do this in order to prove that condition (2) is satisfied. Thus in Case 1 we know that $d=^a c$ and $\hat{c}\neq\emptyset$ and $\widehat{c^*}\neq\emptyset$.

Suppose $e, f \in E$ and $e \subset f$. If $h \in E$ then $h = dg$ or d^*g for some $g \in G$. Suppose $e \subset h \subset f$. Let

$$\hat{h} = \bigcap \{b \in N \mid h \subset^a b\}.$$

Then $\hat{h} \in N$ and $\hat{e} \subset \hat{h} \subset \hat{f}$. Thus there are only finitely many possibilities for $\hat{h}$. Similarly there are only finitely many possibilities for $\widehat{h^*}$.
But

$$h =^a \hat{h} \subset h \subset (\widehat{h^*})^* =^a h$$

by 2.9. Thus there are only finitely many possibilities for h, and so condition (2) is satisfied.

Thus E is the directed edge set of a tree T and there is an action of G on T. It may be the case that for some $g \in G$, $dg = d^*$, in which case T/G will not be a graph. However this difficulty can be overcome by replacing T by its first barycentric subdivision.

Thus we can assume that G acts on T so that T/G has one edge. Let $e \in \delta d$. If d is narrow then it follows from 2.5 that there are only finitely many $b \in E$ for which $e \in \delta b$. Let $eg = e$, i.e. $g \in G_e$. Now $d \cap dg$ and $d^* \cap d^*g$ contain the vertices of e and so cannot be empty. Thus either $d \subset dg$ or $dg \subset d$. Suppose $dg \subset d$ but $dg \neq d$. If d is narrow then we have a contradiction since dg^m, $m = 1, 2, \ldots,$ are infinitely many distinct narrow cuts and $e \in \delta(dg^m)$. If we are dealing with Case 1 (so that d is possibly not narrow), then $d =^a d_1$, $d \subset d_1$ where $d_1 = (\widehat{c^*})^*$. Also since $cg \subset^a c$ it follows that $d_1 g \subset d_1$, $d_1 g \neq d_1$. But d_1 is narrow and the intersection of the $d_1 g^m$ contains the intersection of the dg^m which contains a vertex of e. This contradicts 2.2. Thus $dg = d$ and so $g \in G_d$. An element of G_d must permute the elements of δd and so the index of G_e in G_d is finite. This completes the proof of Theorem 4.1. ∎

Corollary 4.2. *Let the group G act on the graph Γ and suppose $e(\Gamma) > 1$ then either*

$$G = A *_C B$$

or G is an HNN-group

$$G = gp(A, t \mid t^{-1}Ct = D)$$

where C, D are isomorphic subgroups of A, and in either case the subgroup C contains the stabilizer of a directed edge of Γ as a subgroup of finite index.

Proof. This follows easily from Theorem 4.1 using the Bass—Serre theory (see [2, p. 14]). ∎

The decomposition is proper if and only if G does not stabilize a vertex of T. Now T/G has one edge. If there is a path in T of length at least three then there exists $g \in G$ and a directed edge $e \in ET$ such that $eg > e$. Thus by [2, p. 28] G does not stabilize a vertex of T. If Γ/G is finite, then using the techniques of 2.6 it can be shown that there exists $g \in G$ for which d is properly contained in dg or d^*g. Also there exists $g_1 \in G$ for which d^*g_1 or dg_1 is properly contained in d. It follows that the diameter of T is at least three. Thus *the decomposition of G induced by the action of G on T is proper if Γ/G is finite.*

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Added in proof. The author has recently noticed that the following two papers contain results which have non-empty intersection with the results of this paper.

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